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INTEGRATION HAVING VALUES IN A BANACH SPACE

by

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CONTENTS

ABSTRACT

of

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The paper is a report on some work of Bartle, Dunford and Schwartz on a generalization of the Lebesgue Integral and also includes some related material. Much of the work of Bartle, Dunford and Schwartz may be found in their paper Weak Compactness and Vector Measures.

In the first part of the paper, a measure space (S,Σ) with the positive measure ν defined on it is discussed and a number of its properties are derived. In particular, it can be considered a metric space.

Then the properties of the space of measures which are of finite variation on this space are discussed. It is shown that this space is a Banach Space. Equivalent conditions for the the conditional weak compactness of subsets of this space are considered.

In the final part of the paper, a generalized Lebesgue integration is defined and many of the ususal properties of the Lebesgue integral are shown to hold, the final result being the bounded convergence theorem.

Introduction

The following is a report on some work of Bartle, Dunford and Schwartz [1] on a generalization of the Lebesgue integral and also includes some related material. The author in writing this paper has three aims in mind.

First, difficult points in the text are explained and, in many cases,these points are listed as separate lemmas.

Secondly, in a number of places, definitions have been changed and, as a result, a number of changes in material was introduced. It is felt by the author that these changes made the approcach to the problem both clearer and more consistent. In particular, a slight modification was made in the definition of a simple function. This resulted in a subtle but important change in the definition of integration of these functions. This change, however, introduced rigor into the proofs of the subsequent theorems.

Finally, collateral material was included in the paper in order to obtain a unified approach to the problem. In this manner only the most well knowned theorems of analysis were left unproved.

In his book, *Lecons sur l'integration*, Lebesgue [7] attempts to define an integration process yielding a real number for any function f which is in the class of bounded real valued functions and any finite interval [a,b] with the following conditions:

1.) For a, b, h $\int_a^b f(s)ds = \int_{a+h}^{b+h} f(s-h)ds$.

2.) For all a, b, c $\int_a^b f + \int_b^c f + \int_c^a f = 0$.

3.) For all f_1 and f_2 $\int(f_1 + f_2) = \int f_1 + \int f_2$.

4.) $f \geqq 0$ and $b>a$ implies that $\int_a^b f \geqq 0$.

5.) $\int_0^1 1 \cdot ds = 1$.

6.) If $f_n(s) \leqq f_{n+1}(s)$ for all n,s and $\lim\limits_{n\to\infty} f_n(s) = f(s)$ for all s, then $\lim\limits_{n\to\infty} \int f_n = \int f$.

Lebesgue was able to solve the problem but only after he further restricted both the class of functions and the domain of integration. He accomplished this by considering the class of measurable subsets of the straight line, a measure function of these sets and a class of measurable functions.

The Lebesgue integral thus defined, leads quite naturally to a number of extensions and generalizations. For instance, one would expect to be able to define the functions on an n-dimensional rectangle rather than on an interval. This extension turned out to be quite easily accomplished. Following this lead, one might next consider defining the integrability and the measurability of functions from an n-dimensional interval to a Banach Space. This was done by Bochner [2] and it proved quite successful.

Along these lines, Bartle, Dunford and Schwartz [1] further generalized the idea of Lebesgue integration. In their paper, they consider a space S of measurable sets and the measure with respect to which the sets are measurable is from S to a Banach Space X. For measurable functions they consider scalar functions defined on S such that the inverse of a Borel set in the scalars is measurable. That the functions are essentially bounded with respect to this measure is sufficient to guarantee integrability.

In the first chapter of this paper some of the properties of countably additive measures are derived. We have particular interest in the space of measures which are of finite variation on a measure space (S,Σ). For an arbitrary sequence in this space we show that there is a positive measure of finite variation such that the members of the sequence are absolutely continuous.

In the second chapter, we show that the measure space (S,Σ) with a positive measure defined on it may be considered as a metric space. Also a form of the Vitali-Hahn-Saks Theorem is proved.

In the third section we show that the space of measures which are of finite variation is a Banach Space and we develop equivalent conditions for the conditional weak compactness of certain subsets of this Banach Space.

In the final chapter, integration is defined and many of the usual properties of Lebesgue integration are shown to hold and the last of these is the bounded convergence theorem. It should be noted that the material in the third chapter is crucial for the proofs of the theorems in Chapter 4.

In the latter part of the paper of Bartle, Dunford and Schwartz, which is not treated here, an application of the integration theory developed is considered. They present a "vector" generalization of the Riesz Representation Theorem and obtain representation theorems for the general, the weakly compact and the compact operators on a space of continuous functions.

Chapter I

COUNTABLY ADDITIVE MEASURES AND THEIR PROPERTIES

Before proceeding with the central material of this paper, I will make some preliminary definitions and state some standard results, the proofs of which can be found in the appendix.

1.1 Definition. Let S be a set and Σ a collection of subsets of S with the following properties:

1.) Σ is non-empty.

2.) $\phi \varepsilon \Sigma$.

3.) If A is in Σ, then S-A is in Σ.

4.) $A_1, A_2, \ldots, A_n \varepsilon \Sigma \Rightarrow \bigcup_{i=1}^{n} A_i \ \varepsilon \Sigma$.

Then Σ is called a field of subsets of S.

By a simple application of De Morgan's Law, Σ is closed under finite intersections. Further, it follows from the definition that $S \varepsilon \Sigma$ and the difference of two sets in Σ is again in Σ. Finally, the symmetric difference of two sets $A, B \varepsilon \Sigma$ defined by $A \Delta B = (A \cap \complement B) \cup (B \cap \complement A)$ is in Σ.

1.2 Definition. A field, Σ, of subsets of S is called a σ-field if it is closed under countable unions.

Again, by an application of De Morgan's Law, it is seen that a σ-field is closed under countable intersections.

1.3 Definition. A measurable space, (S,Σ), is a set S along with Σ, a σ-field of subsets of S.

1.4 Definition. μ, (an extended real-valued or) complex-valued set function defined for sets of Σ of the measurable space (S,Σ) is called a (countably additive) measure if the following conditions hold:

1.) When μ is real-valued, μ assumes at most one of the values $\pm\infty$.

2.) $\mu(\phi) = 0$.

3.) $\mu(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mu(E_i)$ for every disjoint sequence of E_i in Σ.

1.5) Definition. A measure μ on a measurable space (S,Σ) is called a positive measure if $\mu(A)$ is real-valued and positive or zero for all A in Σ.

1.6 Definition. Let μ be a measure defined on (S,Σ). For E in Σ, define the total variation of μ on E by

$v(\mu,E) = \text{Sup}\ (\sum_{i=1}^{n} |\mu E_i|)$ where $E_i \varepsilon \Sigma$, $E_i \subseteq E$ and for $i \neq j$ $E_i \cap E_j = \phi$.

1.7 Definition. We say that μ, a measure defined on (S,Σ), is of finite variation if $v(\mu,S) < \infty$.

Throughout the remainder of the paper, the following notation will be used consistently:

1.) $Ca(\Sigma) = \{\mu: \mu$ real or complex valued measure on S having finite variation$\}$

2.) Generic elements of $Ca(\Sigma)$ will be called μ.

3.) A positive element of $Ca(\Sigma)$ will be called ν.

4.) $|\lambda|(E)$ is the total variation over E in Σ of a measure λ.

5.) $|\lambda|$ is the total variation of λ on S.

1.8 Lemma. If $\mu \in Ca(\Sigma)$ and $A, B \in \Sigma$, then

$$|\mu|(A \cup B) = |\mu|(A) + |\mu|(B).$$

Proof: Let $E_1, E_2, \ldots, E_n \in \Sigma$ such that $E_i \subseteq A \cup B$ for $i=1,2,\ldots,n$. Then, since $E_i \cap A,\ E_i \cap B \in \Sigma$

$$\begin{aligned}\sum_{i=1}^{n} |\mu(E_i)| &= \sum_{i=1}^{n} |\mu((A \cup B) \cap E_i)| \\ &= \sum_{i=1}^{n} |\mu(A \cap E_i) + \mu(B \cap E_i)| \\ &\leq \sum_{i=1}^{n} |\mu(A \cap E_i)| + \sum_{i=1}^{n} |\mu(B \cap E_i)| \\ &\leq |\mu|(A) + |\mu|(B).\end{aligned}$$

Thus, $|\mu|(A \cup B) \leq |\mu|(A) + |\mu|(B)$.

Conversely, for $\varepsilon > 0$ choose $E_1, E_2, \ldots, E_n \subseteq A$, and $F_1, F_2, \ldots, F_m \subseteq B$ such that $E_i, F_j \in \Sigma$ for $i=1,\ldots,n$ and $j=1,\ldots,m$ and such that

$$|\mu|(A) < \sum_{i=1}^{n} |\mu E_i| + \varepsilon$$

$$|\mu|(B) < \sum_{j=1}^{m} |\mu F_j| + \varepsilon$$

Therefore $$\begin{aligned}|\mu|(A) + |\mu|(B) &< \sum_{i=1}^{n} |\mu(E_i)| + \sum_{j=1}^{m} |\mu(F_j)| + 2\varepsilon \\ &< |\mu|(A \cup B) + 2\varepsilon\end{aligned}$$

and we have $|\mu|(A) + |\mu|(B) = |\mu|(A \cup B)$

Q. E. D.

1.9 Corollary. If $E_1, E_2, \ldots, E_n \in \Sigma$, $\mu \in Ca(\Sigma)$, then

$$|\mu|\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} |\mu|(E_i)$$

1.10 Corollary. If $A,B\varepsilon\Sigma$, $\mu\varepsilon Ca(\Sigma)$, then $|\mu|(A\cup B) \leq |\mu|(A)$.

1.11 Lemma. Let (S,Σ) be a measurable space and let $\mu\varepsilon Ca(\Sigma)$. Then, if $\{A_i\}$ is a sequence of disjoint sets in Σ,

$$|\mu|(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} |\mu|(A_i).$$

Proof: Let $E_1,E_2,\ldots,E_n \varepsilon\Sigma$ be disjoint such that $E_j \subseteq \bigcup_{i=1}^{\infty} A_i$ for $j=1,2,\ldots,n$. Noting that $A_i \cap E_j \varepsilon \Sigma$,

$$\begin{aligned}
\sum_{j=1}^{n} |\mu(E_j)| &= \sum_{j=1}^{n} |\mu(E_j \cap \bigcup_{i=1}^{\infty} A_i)| \\
&= \sum_{j=1}^{n} |\mu(\bigcup_{i=1}^{\infty} A_i \cap E_j)| \\
&= \sum_{j=1}^{n} |\sum_{i=1}^{\infty} \mu(A_i \cap E_j)| \\
&\leq \sum_{j=1}^{n} \sum_{i=1}^{\infty} |\mu(A_i \cap E_j)| \\
&\leq \sum_{i=1}^{\infty} \sum_{j=1}^{n} |\mu(A_i \cap E_j)| \\
&\leq \sum_{i=1}^{\infty} |\mu|(A_i).
\end{aligned}$$

Thus $|\mu|(\bigcup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} |\mu|(A_i)$.

On the other hand, for $N>0$ let $E_{i,1},E_{i,2},\ldots,E_{i,n_i} \subseteq A_i$ $i=1,2,\ldots,N$ be disjoint sets of Σ. Then, since the A_i are disjoint, $E_{1,1},\ldots,E_{1,n_1},\ldots,E_{N,1},\ldots,E_{N,n_N}$ are disjoint sets of Σ such that

$$E_{i,j} \subseteq \bigcup_{k=1}^{N} A_k \subseteq \bigcup_{k=1}^{\infty} A_k \quad \text{for } j=1,2,\ldots,n_i,\ i=1,2,\ldots,N.$$

Thus $\sum_{i=1}^{N} \sum_{j=1}^{n_i} |\mu(E_{i,j})| \leq |\mu|(\bigcup_{i=1}^{\infty} A_i)$

$$\Rightarrow \sum_{i=1}^{N} |\mu|(A_i) \leq |\mu|(\bigcup_{i=1}^{\infty} A_i) \quad \text{for all } N$$

$$\Rightarrow \sum_{i=1}^{\infty} |\mu|(A_i) \leq |\mu|(\bigcup_{i=1}^{\infty} A_i).$$

Q. E. D.

1.12 Lemma. If $\{\lambda_i\}$ is a sequence in $Ca(\Sigma)$, then there is a positive $\nu\varepsilon Ca(\Sigma)$ such that $\lim\limits_{\nu(E)\to 0} \lambda_i(E) = 0$ for $i=1,2,\ldots$

Proof: Let ν be defined by

$$\nu(E) = \sum_{n=1}^{\infty} |\lambda_n|(E)[2^n(1+|\lambda_n|)]^{-1} \quad \text{for } E\varepsilon\Sigma.$$

Since $|\lambda_n|(\phi) = 0$, $\nu(\phi) = 0$. Now if $\{A_i\}$ is a sequence of disjoint sets in Σ, we see from lemma 1.11 that

$$\begin{aligned}\nu(\bigcup_{n=1}^{\infty} A_i) &= \sum_{n=1}^{\infty} |\lambda_n|(\bigcup_{n=1}^{\infty} A_i)[2^n(1+|\lambda_n|)]^{-1} \\ &= \sum_{n=1}^{\infty} [\sum_{i=1}^{\infty} |\lambda_n|(A_i)][2^n(1+|\lambda_n|)]^{-1} \\ &= \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} |\lambda_n|(A_i)[2^n(1+|\lambda_n|)]^{-1} \\ &= \sum_{i=1}^{\infty} \nu(A_i).\end{aligned}$$

Thus ν is countably additive and hence a measure. But clearly $\nu(E)\geq 0$ for $E\varepsilon\Sigma$, and we see that ν is a positive measure. Finally, we show that ν is of finite variation. Let $E_1, E_2, \ldots, E_n$ be a sequence of disjoint sets in Σ such that $E_i \subseteq E$ for $i=1,2,\ldots,n$. Then, since ν is a positive measure,

$$\sum_{i=1}^{n} |\nu(E_i)| = \sum_{i=1}^{n} \nu(E_i)$$

$$= \nu\left(\bigcup_{i=1}^{n} E_i\right)$$

$$\leq \nu(E)$$

$$\leq \sum_{n=1}^{\infty} |\lambda_n|(E)[2^n(1 + |\lambda_n|)]^{-1}$$

$$\leq \sum_{n=1}^{\infty} 2^{-n}$$

≤ 1 and thus $|\nu|(E) \leq 1$ for all $E \varepsilon \Sigma$ and thus $|\nu| \leq 1$. Therefore $\nu \varepsilon Ca(\Sigma)$.

Now let $\varepsilon > 0$ be given and fix n. Choose $|\nu(E)| < \varepsilon[2^n(1 + |\lambda_n|)]^{-1}$.

Then $\left|\sum_{j=1}^{\infty} |\lambda_j|(E)[2^j(1 + |\lambda_j|)]^{-1}\right| < \varepsilon[2^n(1 + |\lambda_n|)]^{-1}$ implies

that $|\lambda_n|(E)[2^n(1 + |\lambda_n|)]^{-1} < \varepsilon[2^n(1 + |\lambda_n|)]^{-1}$

$$\Longrightarrow |\lambda_n|(E) < \varepsilon$$

But $|\lambda_n(E)| \leq |\lambda_n|(E) < \varepsilon \Longrightarrow \lim_{\nu(E)\to 0} \lambda_n(E) = 0$.

Q. E. D.

Chapter II

THE METRIC SPACE Σ AND THE VITALI-HAHN-SAKS THEOREM

2.1 Definition. Let λ,μ be set functions defined on a σ-field Σ. The we call λ μ-continuous if

$$\lim_{v(\mu,E)\to 0} \lambda(E) = 0.$$

2.2 Theorem. Let (S,Σ) be a measurable space with the positive measure ν defined on it. Then Σ may be considered as a complete metric space.

Proof: Let '$\simeq$' be the equivalence defined on Σ by setting $A\simeq B$ if $\nu(A\Delta B) = 0$. If we let '[ ]' denote the equivalence classes under $\simeq$, then we may define d by

$$d([A],[B]) = \nu(A\Delta B).$$

First, I will show that d is well defined. Thus let $A\simeq A'$ and $B\simeq B'$. Since ν is positive, for $E\subseteq A\Delta A'$ $\nu(E) = 0$. Thus

$$\begin{aligned}
\nu(A\Delta B) &= \nu(A-B) + \nu(B-A) \\
&= \nu((A-B)\cap(A'\cup \complement A')) + \nu((B-A)\cap(A'\cup\complement A')) \\
&= \nu[(A\cap\complement B\cap A')\cup(A\cap\complement A'\cap\complement B)] + \nu[(B\cap\complement A\cap A')\cup(B\cap\complement A\cap\complement A')] \\
&= \nu[A\cap\complement B\cap A'] + \nu[B\cap\complement A\cap\complement A'] \\
&= \nu[(A'\cap\complement B\cap A)\cup(A'\cap\complement B\cap\complement A)] + \nu[(B\cap\complement A'\cap\complement A)\cup(B\cap\complement A'\cap A)] \\
&= \nu(A'-B) + \nu(B-A') \\
&= \nu(A'\Delta B)
\end{aligned}$$

However, $\nu(A\Delta B) = \nu(B\Delta A)$ and therefore

$$\nu(A\Delta B) = \nu(A'\Delta B) = \nu(B\Delta A') = \nu(B'\Delta A') = \nu(A'\Delta B').$$

Thus d is well defined and we will henceforth write $d([A],[B])$ as $d(A,B)$ without ambiguity. Now I will show that d is a metric. To verify that d satisfies the four properties of a metric, let $A,B,C\in\Sigma$.

1.) Since ν is positive, $d(A,B) = \nu(A\Delta B) \geq 0$.

2.) $d(A,B) = 0$ if and only if $\nu(A\Delta B) = 0$ if and only if $A\simeq B$.

3.)
$$\begin{aligned} d(A,B) &= \nu(A\Delta B) \\ &= \nu(A\cap \complement B) + \nu(B\cap \complement A) \\ &= \nu(A\cap\complement B\cap C) + \nu(A\cap\complement B\cap\complement C) + \nu(B\cap\complement A\cap C) \\ &\qquad + \nu(B\cap\complement A\cap\complement C) \\ &\leq \nu(A\cap C) + \nu(A\cap\complement C) + \nu(B\cap C) + \nu(B\cap\complement C) \\ &\leq d(A,C) + d(C,B). \end{aligned}$$

4.) $d(A,B) = \nu(A\Delta B) = \nu(B\Delta A) = d(B,A)$.

Thus d is a metric.

Finally, I show d is a complete metric. Let $\{C_i\}\subseteq\Sigma$ be a Cauchy sequence. Choose K_n such that $K_n > K_{n-1}$ and such that $p,q > K_n \Rightarrow \nu(C_p\Delta C_q) < 2^{-n}$. Let $B_i = C_{K_i}$. Then $\{B_i\}$ is a Cauchy sequence and $\nu(B_{n+1}\Delta B_n) < 2^{-n}$ for all n. If we set $A_n = B_{n+1} - B_n$, then

$$\nu(A_n) = \nu(B_{n+1}-B_n) = \nu(B_{n+1}\Delta B_n) < 2^{-n} \quad \text{for } n=1,2,\ldots.$$

Now $B_n \cup \bigcup_{i=n}^{\infty} A_i = \bigcup_{i=n}^{\infty} B_i$. Thus

$$\begin{aligned} \nu\left(\bigcup_{i=n}^{\infty} B_i - B_n\right) &= \nu\left[\left(B_n \cup \bigcup_{i=n}^{\infty} A_i\right) - B_n\right] \\ &= \nu\left(\bigcup_{i=n}^{\infty} A_i\right) \\ &= \sum_{i=n}^{\infty} \nu(A_i) = \sum_{i=n}^{\infty} 2^{-i} < 2^{-n+1} \quad \text{for } n=1,2,\ldots \end{aligned}$$

Let $B = \bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty}$ and we see that $B\varepsilon\Sigma$. Now

$$\begin{aligned} \nu(B) &\leqq \nu(\bigcup_{n=1}^{\infty} B_n) \\ &\leqq \nu(\bigcup_{n=1}^{\infty} B_n - B_1) + \nu(B_1) \\ &< 1 + \nu(B_1) \end{aligned}$$

Also I make note of the fact that $\{\bigcup_{i=n}^{\infty} B_i\}$ is a decreasing sequence and as a result

$$\begin{aligned} \nu(B) &= \nu(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} B_i) \\ &= \underset{n\to\infty}{\text{limit}}\ \nu(\bigcup_{i=n}^{\infty} B_i). \end{aligned}$$

Thus, for $\varepsilon>0$, choose N such that $n>N$ implies that

$\nu(\bigcup_{i=n}^{\infty} B_i - B) < \varepsilon/2$ and $\nu(\bigcup_{i=n}^{\infty} B_i - B_n) < \varepsilon/2$. Hence, for $n>N$

$$\begin{aligned} \nu(B_n \Delta B) &\leqq \nu(B_n \Delta \bigcup_{i=n}^{\infty} B_i) + \nu(B \Delta \bigcup_{i=n}^{\infty} B_i) \\ &\leqq \nu(\bigcup_{i=n}^{\infty} B_i - B_n) + \nu(\bigcup_{i=n}^{\infty} B_i - B) < \varepsilon \end{aligned}$$

Consequently, $\underset{n\to\infty}{\text{limit}}\ B_n = B$. But $\{B_n\}$ is a subsequence of the Cauchy sequence $\{C_i\}$ and thus $\underset{n\to\infty}{\text{limit}}\ C_n = B$. Therefore, (Σ,d) is a complete metric space.

Q. E. D.

2.3 Lemma. Let ν be as in Theorem 2.2. Then λ is continuous in (Σ,d) if and only if λ is ν-continuous in (S,Σ).

Proof: Since ν is positive, I first note that $v(\nu,E) = \nu(E)$. For let $A_1, A_2, \ldots, A_n \in \Sigma$ be disjoint sets such that $A_i \subseteq E$ for $i=1,2,\ldots,n$. Then

$$\sum_{i=1}^{n} |\nu(E_i)| = \sum_{i=1}^{n} \nu(E_i) = \nu(\bigcup_{i=1}^{n} E_i) \leqq \nu(E) \leqq v(\nu,E).$$ Thus

$v(\nu,E) \leqq \nu(E) \leqq v(\nu,E)$. Therefore, we need only show that λ is ν-continuous if and only if $\lim_{\nu(E)\to 0} \lambda(E) = 0$.

First suppose λ is ν-continuous and let $\varepsilon>0$ be given. Choose δ such that $\nu(E)<\delta \Rightarrow |\lambda(E)| < \varepsilon/2$. But if $d(E,F) < \delta$, then $\nu(F-E) < \delta$ and $\nu(E-F) < \delta$ and this in turn means that $|\lambda(F-E)| < \varepsilon/2$ and $|\lambda(E-F)| < \varepsilon/2$. Thus

$$\begin{aligned} |\lambda(E)-\lambda(F)| &= |\lambda(E-F) + \lambda(E\cup F) - \lambda(F-E) - \lambda(F\cup E)| \\ &\leqq |\lambda(E-F)| + |\lambda(F-E)| \\ &< \varepsilon \end{aligned}$$

and λ is continuous.

Conversely, if λ is continuous, let $\varepsilon>0$ be given and choose δ such that $d(E,F)<\delta \Rightarrow |\lambda(E)-\lambda(F)| < \varepsilon$. In particular, if $d(E,\phi) = \nu(E) < \delta$, then $|\lambda(E)-\lambda(\phi)| = |\lambda(E)| < \varepsilon$ and λ is ν-continuous.

Q. E. D.

Now we are prepared to prove a special form of the Vitali-Hahn-Saks Theorem which will be useful for our purposes.

2.4 <u>Theorem</u>. Let ν be a positive measure on Σ, let $\{\lambda_n\} \subseteq \Sigma$ be a sequence such that $\lim_{n\to\infty} \lambda_n(E)$ exists for all $E \in \Sigma$ and let $\lim_{v(\nu,E)\to 0} \lambda_n(E) = 0$ for $n=1,2,\ldots$. Then this limit is uniform in n.

<u>Proof</u>: By lemma 2.3, each λ_n is continuous. Thus, for $\varepsilon>0$, the sets

$$A_{n,m} = \{E\colon E\varepsilon\Sigma \text{ and } |\lambda_n(E)-\lambda_m(E)| \leq \varepsilon\} \quad n,m=1,2,\ldots$$

and $B_j = \bigcap_{n,m\geq j} A_{n,m}$ are closed in the complete metric space (Σ,d).

For $E\varepsilon\Sigma$, $\lim_{n\to\infty} \lambda_n(E)$ exists and therefore there is an N such that $n,m\geq N \Rightarrow |\lambda_n(E)-\lambda_m(E)| < \varepsilon$. Hence $E\varepsilon A_{n,m}$ for $n,m\geq N$ implies that $E\varepsilon B_n$, and thus $(\Sigma,d) = \bigcup_{j=1}^{\infty} B_j$. By the Baire Category Theorem, at least one of the B_j has non-empty interior. Thus there exist $F\varepsilon B_j$ and $r>0$ such that

$$K = \{E\colon E\varepsilon\Sigma,\ d(E,F)<r\}\subseteq B_j$$

and as a result

$|\lambda_n(E)-\lambda_m(E)| < \varepsilon$ for all $n,m\geq j$ for $E\varepsilon K$. Choose $\delta<r$ such that $|\lambda_n(B)| < \varepsilon$ for $n=1,2,\ldots,j$ for all $B\varepsilon\Sigma$ when $\nu(B) < \delta$. But $\nu(B) < \delta$ implies that

$$d(B\cup F,F) = \nu(B\cup F\Delta F) \leq \nu(B) < \delta < r \text{ and}$$

$$d(F-B,F) = \nu(F-B\Delta F) \leq \nu(B) < \delta < r$$ and we see that $F\cup B$ and $F-B$ are contained in K. Now $\lambda_n(F-B) + \lambda_n(B) = \lambda_n(F\cup B)$ which implies that $\lambda_n(B) = \lambda_n(F\cup B) - \lambda_n(F-B)$ and for $n>j$

$$\begin{aligned}
|\lambda_n(B)| &= |\lambda_j(B)+\lambda_n(B)-\lambda_n(B)| \\
&= |\lambda_j(B)+\lambda_n(B\cup F)-\lambda_n(F-B)+\lambda_j(F-B)-\lambda_j(B\cup F)| \\
&= |\lambda_j(B)|+|\lambda_n(B\cup F)-\lambda_j(B\cup F)|+|\lambda_n(F-B)-\lambda_j(F-B)| \\
&< 3\varepsilon
\end{aligned}$$

Hence, we have $|\lambda_n(B)| < 3\varepsilon$ for $n=1,2,\ldots$

Q. E. D.

2.5 Corollary. If the hypotheses of Theorem 2.4 are satisfied and in addition $\lambda_i \varepsilon Ca(\Sigma)$ for $i=1,2,...$ then λ defined on Σ by

$$\lambda(E) = \lim_{n\to\infty} \lambda_n(E) \quad \text{for } E\varepsilon\Sigma$$

is in $Ca(\Sigma)$.

Proof: Clearly $\lambda(\phi) = 0$. To see that λ is countably additive, let $S = A\cup B$ be the Hahn decomposition of S with respect to λ where $A\cap B = \phi$ and $\lambda(A)\geq 0$ and $\lambda(B)\leq 0$. If $\{E_i\}\subseteq\Sigma$ is disjoint, then $\{\bigcup_{j=n}^{\infty} E_j\cap A\}$ and $\{\bigcup_{j=n}^{\infty} E_j\cap B\}$ are sequences decreasing to the void set and we conclude that

$$\lim_{n\to\infty} \nu(\bigcup_{j=n}^{\infty} E_j\cap A) = \lim_{n\to\infty} \nu(\bigcup_{j=n}^{\infty} E_j\cap B) = 0.$$

Thus from Theorem 2.4

$$\lim_{n\to\infty} \lambda_k(\bigcup_{i=n}^{\infty} E_i\cap A) = 0 \quad \text{and} \quad \lim_{n\to\infty} \lambda_k(\bigcup_{i=n}^{\infty} E_i\cap B) = 0,$$

with the limits both uniform in k. Therefore, choose N such that $n\geq N$ implies that

$$|\lambda_k(\bigcup_{i=n}^{\infty} E_i\cap A)| < \varepsilon/2 \quad \text{and} \quad |\lambda_k(\bigcup_{i=n}^{\infty} E_i\cap B)| < \varepsilon/2 \text{ for } k=1,2,...$$

First I observe that λ is finitely additive, for

$$\lambda(\bigcup_{i=1}^{n} E_i) = \lim_{k\to\infty} \lambda_k(\bigcup_{i=1}^{n} E_i) = \lim_{k\to\infty} \sum_{i=1}^{n} \lambda_k(E_i)$$

$$= \sum_{i=1}^{n} \lim_{k\to\infty} \lambda_k(E_i) = \sum_{i=1}^{n} \lambda(E_i).$$

Thus, since the E_i are disjoint,

$$|\lambda(\bigcup_{i=1}^{\infty} E_i) - \sum_{i=1}^{n-1} \lambda(E_i)| = |\lambda(\bigcup_{i=n}^{\infty} E_i)|$$

and $\left|\lambda\left(\bigcup_{i=n}^{\infty} E_i\right)\right| \leq \left|\lambda\left(\bigcup_{i=n}^{\infty} E_i \cap A\right)\right| + \left|\lambda\left(\bigcup_{i=n}^{\infty} E_i \cap B\right)\right| < \varepsilon$

which means that $\sum_{i=1}^{\infty} \lambda(E_i) = \lambda\left(\bigcup_{i=1}^{\infty} E_i\right)$ and λ is countably additive.

Finally, I show that λ is of finite variation. Let S have the same decomposition as above. If $E_1, E_2, \ldots, E_n \varepsilon \Sigma$ are disjoint and $E_i \subseteq A$ for $i=1,2,\ldots,n$, then

$$\begin{aligned}\sum_{i=1}^{n} |\lambda(E_i)| &= \left|\sum_{i=1}^{n} \lambda(E_i)\right| \\ &= \left|\lambda \bigcup_{i=1}^{n} E_i\right| \\ &\leq |\lambda(A)|.\end{aligned}$$

Thus $v(\lambda,A) \leq |\lambda(A)| \leq v(\lambda,A)$ and we have $v(\lambda,A) = |\lambda(A)|$. Similarly, $v(\lambda,B) = |\lambda(B)|$. Therefore, choose N such that $|\lambda(A)-\lambda_N(A)| < 1$ and $|\lambda(B)-\lambda_N(B)| < 1$. Thus, if $E_1, E_2, \ldots, E_n$ are disjoint subsets in Σ

$$\begin{aligned}\sum_{i=1}^{n} |\lambda(E_i)| &\leq \sum_{i=1}^{n} |\lambda(E_i \cap A)| + \left|\sum_{i=1}^{n}\right| \lambda(E_i \cap B)| \\ &\leq v(\lambda,A) + v(\lambda,B) \\ &< |\lambda_N(A)| + |\lambda_N(B)| + 2 < \infty\end{aligned}$$

Q. E. D.

Chapter III

WEAK COMPACTNESS

I will first prove some preliminary results about $Ca(\Sigma)$.

3.1 Lemma. $Ca(\Sigma)$ is a linear space.

Proof: For λ and μ in $Ca(\Sigma)$, it is quite clear that $a\lambda + b\mu$ is a countably additive measure. Hence we need only show that $a\lambda + b\mu$ is of finite variation and then it would be a member of $Ca(\Sigma)$. Since λ and μ are of finite variation, $M = |a||\lambda| + |b||\mu|$ is finite. Then, if $E_1, E_2, \ldots, E_n \in \Sigma$ is a disjoint sequence,

$$\begin{aligned}\sum_{i=1}^{n} |(a\lambda+b\mu)(E_i)| &\leq \sum_{i=1}^{n} |a||\lambda(E_i)| + \sum_{i=1}^{n} |b||\mu(E_i)| \\ &\leq |a||\lambda| + |b||\mu| \\ &\leq M.\end{aligned}$$

This implies that $|a\lambda+b\mu| \leq M < \infty$.

Q. E. D.

3.2 Lemma. $Ca(\Sigma)$ is a normed space.

Proof: For $\lambda \in Ca(\Sigma)$ define $||\lambda|| = |\lambda|$

1.) To see that $||\alpha\lambda|| = |\alpha| \cdot ||\lambda||$, observe that

$$\sum_{i=1}^{n} |\alpha\lambda(E_i)| = \sum_{i=1}^{n} |\alpha||\lambda(E_i)| = |\alpha|[\sum_{i=1}^{n} |\lambda(E_i)|]$$

Thus $|\alpha\lambda| = |\alpha||\lambda|$ which means that $||\alpha\lambda|| = |\alpha| \cdot ||\lambda||$.

2.) To see that $||\lambda+\mu|| \leq ||\lambda|| + ||\mu||$ we simply note that $|(\lambda+\mu)(E)| \leq |\lambda(E)| + |\mu(E)|$.

3.) Finally $||\lambda|| = 0$ if and only if $|\lambda(E)| = 0$ for all $E \in \Sigma$ if and only if $\lambda(E) = 0$ for all $E \in \Sigma$ if and only if $\lambda = 0$.

Q. E. D.

3.3 Lemma. $Ca(\Sigma)$ is a complete space.

Proof: Let $\{\lambda_i\} \subseteq Ca(\Sigma)$ be a Cauchy sequence. Then $\{\lambda_i(E)\}$ is a Cauchy sequence of real (complex) numbers for all $E\varepsilon\Sigma$. For, let $\varepsilon>0$ be given and choose N such that p,q>N implies that $|\lambda_p-\lambda_q| < \varepsilon$. Then

$$|\lambda_p(E)-\lambda_q(E)| = |(\lambda_p-\lambda_q)(E)| \leqq |\lambda_p-\lambda_q|(E) \leqq |\lambda_p-\lambda_q| < \varepsilon.$$

Since $\{\lambda_i(E)\}$ is a Cauchy sequence in the real (complex) numbers, $\lim_{i\to\infty} \lambda_i(E)$ exists for all $E\varepsilon\Sigma$ and I am therefore able to define the measure λ as follows:

$$\lambda(E) = \lim_{i\to\infty} \lambda_i(E)$$

Frome Lemma 1.12, there is a positive $\nu\varepsilon Ca(\Sigma)$ such that $\lim_{\nu(E)\to 0} \lambda_i(E) = 0$ for i=1,2,... By Theorem 2.4 and Corollary 2.5, the limit is uniform in i and $\lambda\varepsilon Ca(\Sigma)$.

Suppose that for the moment μ is a real valued measure and that S has the Hahn Decomposition with respect to μ, $S=A\cup B$. Then

$$\sum_{i=1}^{n} |\mu(E_i)| \leqq \sum_{i=1}^{n} \mu(A\ E_i) - \sum_{i=1}^{n} \mu(B\ E_i)$$

$$\leqq |\mu(A)| + |\mu(B)|$$

$$\leqq 2 \sup_{E\varepsilon\Sigma} |\mu(E)|$$

and this means that $v(\mu,S) \leqq 2 \sup_{E} |\mu(E)|$. Analagously, if μ is complex valued

$$v(\mu,S) \leqq 4 \sup_{E\varepsilon\Sigma} |\mu(E)|.$$

We now use this to show that $\lambda_i \to \lambda$ in $Ca(\Sigma)$. Let $\varepsilon>0$ be given and choose N such that p,q>N implies that

$$|\lambda_p - \lambda_q| < \varepsilon/4 \Longrightarrow |\lambda_p(E) - \lambda_q(E)| < \varepsilon/4 \text{ for all } E\varepsilon\Sigma$$

which in turn implies that $|\lambda_p(E) - \lambda(E)| \leqq \varepsilon/4$ for all $E\varepsilon\Sigma$. Thus $|\lambda_p - \lambda| \leqq 4\sup_E |(\lambda_p - \lambda)(E)| \leqq \varepsilon$.

Hence $\lambda_i \to \lambda$ in $Ca(\Sigma)$.

Q. E. D.

In light of the above three lemmas we have that $Ca(\Sigma)$ is a Banach Space and I now state this formally as a theorem.

3.4 <u>Theorem</u>. $Ca(\Sigma)$ is a Banach Space.

We need a number of theorems from classical analysis and then we will be able to state necessary and sufficient conditions for the compactness of subsets of $Ca(\Sigma)$.

The first of these theorems is a form of the uniform boundedness principle, a proof of which may be found in the appendix. Of particular interest for the paper is the corollary.

3.5 <u>Theorem</u>. Let Φ be a family of continuous real valued functions on a complete seminormed space which is pointwise bounded. Then Φ is uniformly bounded.

3.6 <u>Corollary</u>. Let S be a set in a seminormed space X such that f is bounded on S for all f in X'. Then S is norm bounded.

<u>Proof</u>: Let $\hat{X}$ be the natural imbedding of X in the second dual of X and let $\hat{S} = \{\hat{x}: x\varepsilon S\}$.

Now, for $f \epsilon X'$, there exists M_f such that

$|f(s)| \leq M_f$ for all s in S and thus

$|\hat{s}(f)| = |f(s)| \leq M_f$ for all $\hat{s}$ in $\hat{S}$. Hence $\hat{S}$ is a pointwise bounded family and Theorem 3.5 applies. There exists M such that $|\hat{s}(f)| \leq M$ for all $\hat{s}$ in $\hat{S}$, for all f in X'. In particular, there exists M' such that $||\hat{s}|| \leq M'$ for all $\hat{s} \epsilon \hat{S}$. Thus $||s|| = ||\hat{s}|| \leq M'$.

Q.E.D.

3.7 <u>Definition</u>. Given the Banach Space X, the <u>weak</u> <u>topology</u> is the weakest topology on X such that all the members of X^* are continuous.

3.8 <u>Definition</u>. A subset A of X is called <u>conditionally</u> <u>compact</u> if A^- is compact.

It is readily seen that A is conditionally compact if and only if A^- is compact in its relative topology.

Due to the lack of space, I state without proof the Eberlein-Ŝmulian Theorem. A proof of this theorem may be found, for example, in "Linear Operators - Part I" (page 430) by Dunford and Schwartz.

3.9 <u>Theorem</u>. A subset A of a Banach Space is weakly sequentially compact if and only if weakly conditionally compact.

Since, by Theorem 3.4, we are able to consider $Ca(\Sigma)$ as a Banach Space, the Eberlein-Ŝmulian Theorem may be applied to $Ca(\Sigma)$. In particular, it is useful in proving the theorems which follow.

3.10 Theorem. If $K \subseteq Ca(\Sigma)$ is weakly conditionally compact, then 1.) K is bounded

2.) If $\{E_i\}$ is a sequence in Σ decreasing to the void set, then $\lim_{i\to\infty} \lambda(E_i) = 0$ uniformly for $\lambda \in K$.

Proof: Let $K \subseteq Ca(\Sigma)$ be weakly conditionally compact. Thus K^- is weakly compact. Therefore, if f is a continuous functional on $Ca(\Sigma)$, then f is continuous in the weak topology and we have that $f(K^-)$ is compact in the reals. But a compact set in the reals is bounded. Since the choice of f was arbitrary, Corollary 3.6 holds and K^- is norm bounded. Thus the first assertion is proved.

Now suppose the second result is not true. That is, assume there exists $\varepsilon > 0$, a sequence $\{E_i\} \subseteq \Sigma$ which decreases to the void set and a sequence $\{\lambda_i\} \subseteq K$ such that

$$|\lambda_n(E_n)| > \varepsilon \qquad \text{for } n=1,2,\ldots$$

Since K is weakly conditionally compact, by Theorem 3.9, K is weakly sequentially compact and thus $\{\lambda_i\}$ converges in the topology. As a result, $\lim_{n\to\infty} \lambda_n(E)$ exists for all E in Σ. Thus as in lemma 1.12, we may construct the positive $\nu \in Ca(\Sigma)$ such that $\lim_{\nu(E)\to 0} \lambda_i(E) = 0$ for $i=1,2,\ldots$ Then, by Theorem 2.4, this limit is uniform in i. Since the E_i decrease to a void set, $\lim_{n\to\infty} \nu(E_n) = 0$. Thus $\lim_{n\to\infty} \lambda_n(E_n) = 0$ which contradicts the original assumption and the result follows.

Q. E. D.

3.11 Theorem. If $K \subseteq Ca(\Sigma)$ is conditionally weakly compact, then there exists positive $\nu \varepsilon Ca(\Sigma)$ such that

$$\lim_{\nu(E)\to 0} \lambda(E) = 0 \quad \text{uniformly for } \lambda \varepsilon K.$$

Proof: First I will show that for arbitrary $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ and a finite subset $\{\lambda_1, \ldots, \lambda_n\} \subseteq K$ such that $|\lambda_i|(E) < \delta(\varepsilon)$ for $i = 1, 2, \ldots, n$ implies that $|\lambda(E)| < \varepsilon$ for $\lambda \varepsilon K$. To see this, suppose the statement is false for some ε. Choose and arbitrary $\lambda_1 \varepsilon K$. Then there exists $E_1 \varepsilon \Sigma$ and $\lambda_2 \varepsilon K$ such that $|\lambda_1(E_1)| < 1/2$ and $|\lambda_2(E_1)| \geqq \varepsilon$.

Now we proceed by induction. Suppose we have obtained the sequences $\{\lambda_1, \lambda_2, \ldots, \lambda_n\} \subseteq K$ and $\{E_1, E_2, \ldots, E_{n-1}\} \subseteq \Sigma$ with the property that

$$|\lambda_i|(E_j) < 2^{-j} \qquad 1 \leqq i \leqq j \leqq n-1$$

and $$|\lambda_{j+1}(E_j)| \geqq \varepsilon \qquad 1 \leqq j \leqq n$$

Then there exists E_n in Σ and λ_{n+1} in K such that $|\lambda_i|(E_n) < 2^{-n}$ for $i \leqq 1 \leqq n$ and $|\lambda_{n+1}(E_n)| \geqq \varepsilon$. Thus, as in the preceeding theorem, we may apply lemma 1.12 and theorem 2.4 to obtain the positive measure $\nu' \varepsilon Ca(\Sigma)$ such that

$$\lim_{\nu'(E)\to 0} \lambda_i(E) = 0 \text{ uniformly in } i.$$

Expressing ν' in the explicit form of lemma 1.12,

$$\begin{aligned} \nu'(E_j) &= \sum_{n=1}^{\infty} |\lambda_n|(E_j)\,[2^{-n}(1+|\lambda_n|)]^{-1} \\ &\leqq \sum_{n=1}^{j} |\lambda_n|(E_j)\,[2^{-n}(1+|\lambda_n|)]^{-1} + \sum_{n=j+1}^{\infty} 2^{-n} \\ &< \sum_{n=1}^{j} 2^{-j}2^{-n} + 2^{-j} < 2^{-j+1} \end{aligned}$$

Now choose N such that $\nu'(E) < 2^{-n}$ $|\lambda_i(E)|<\varepsilon$ for $i=1,2,\ldots$ Thus, since $\nu'(E_{N+1}) < 2^{-N}$, $|\lambda_{N+2}(E_{N+1})| < \varepsilon$ in contradiction to the way in which the sequences were chosen and the assertion is proven.

Now, to prove the theorem, let $\varepsilon>0$ be given. For $n=1,2,\ldots$ let $\delta(2^{-n})$ and $\{\lambda_{np}: 1\leq p\leq P_n,\ n=1,2,\ldots\}\subseteq K$ be such that

$$|\lambda(E)| < 2^{-n} \quad \text{for all} \quad K \quad \text{whenever}$$

$$|\lambda_{np}|(E) < \delta(2^{-n}) \qquad \text{for } p=1,2,\ldots,P_n$$

Define the positive measure $\nu\varepsilon Ca(\Sigma)$ by

$$\nu(E) = \sum_{n=1}^{\infty} 2^{-n} \sum_{p=1}^{P_n} |\lambda_{np}|(E)[P_n(1+|\lambda_{np}|)]^{-1} \qquad \text{for } E\varepsilon\Sigma$$

Choose n such that $2^{-n} < \varepsilon$. Noting that $M = \sup_{\lambda\varepsilon K} |\lambda| < \infty$ since K is bounded, we see that if $\nu(E) < \delta(2^{-n})[2^n P_n(1+M)]^{-1}$ then $2^{-n}\sum_{p=1}^{P_n}|\lambda_{np}|(E)[P_n(1+|\lambda_{np}|)]^{-1} < (2^{-n})[2^n P_n(1+M)]^{-1}$ which implies that $|\lambda_{np}|(E) < \delta(2^{-n})$ for $p=1,2,\ldots,P_n$ and thus $|\lambda(E)| < 2^{-n} < \varepsilon$ for all $\lambda\varepsilon K$ by the first remarks of the proof.

Q. E. D.

The latter part of the above proof yields the stronger corollary:

3.12 Corollary. If $K\subseteq Ca(\Sigma)$ is weakly conditionally compact, then there exists a positive measure $\nu\varepsilon Ca(\Sigma)$ such that

1.) $\lim_{\nu(E)\to 0} \lambda(E) = 0$ uniformly for $\lambda\varepsilon K$.

2.) For $E\varepsilon\Sigma$, $\nu(E)<\varepsilon$ whenever $|\lambda|(E)<\varepsilon$ for all $\lambda\varepsilon K$.

It will shortly be shown that the converse of Theorem 3.10 is also true. But first I will show that the condition of Theorem 3.11 may replace the second condition of Theorem 3.10, thus yielding a corollary to the converse theorem.

3.13 Lemma. Let $K \subseteq Ca(\Sigma)$ and let $\nu \varepsilon Ca(\Sigma)$ be positive such that $\lim_{\nu(E)\to 0} \lambda(E) = 0$ uniformly for $\lambda \varepsilon K$. Then, if $\{E_i\} \subseteq \Sigma$ decreases to the void set,

$$\lim_{i\to\infty} \lambda(E_i) = 0 \text{ uniformly for } \lambda \varepsilon K.$$

Proof: Let $\varepsilon > 0$ be given and choose δ such that $\nu(E) < \delta$ implies that $|\lambda(E)| < \varepsilon$ for all $\lambda \varepsilon K$. Since $\{E_i\}$ decreases to the void set, there exists N such that $n > N \Rightarrow \nu(E_i) < \delta$ and the result follows.

Q. E. D.

The following three lemmas will prove useful in proving the converse of Theorem 3.10. The first two of these are characterizations of fields and σ-fields respectively.

3.14 Lemma. If Ψ is a countable collection of sets, then the field generated by Ψ is countable.

Proof: Without loss of generality, $\phi \varepsilon \Psi$. Let $\Psi_0 = \Psi$ and let Ω be the field generated by Ψ_0. Let

$$\Psi_1 = \{\bigcup_{i=1}^{n} (E_i - F_i) : E_i, F_i \varepsilon \Psi_0\}$$ and in general, let

$$\Psi_n = \{\bigcup_{i=1}^{k} (E_i - F_i) : E_i, F_i \varepsilon \Psi_{n-1}\}.$$ Then $\Psi \subseteq \bigcup_{n=0}^{\infty} \Psi_n \subseteq \Omega$.

The proof will be completed by showing that the countable collection $\bigcup_{n=0}^{\infty} \Psi_n$ is a ring and thus equal to Ω. Hence,

Ω will be countable. It is clear that $\phi\varepsilon\bigcup_{n=0}^{\infty}\Psi_n$ since $\phi\varepsilon\Psi_0$. If $A\varepsilon\bigcup_{n=0}^{\infty}\Psi_n$, then there exists k such that $A\varepsilon\Psi_k$ and thus

$\phi-A\varepsilon\Psi_{k+1}$ which implies that $\complement A\varepsilon\bigcup_{n=0}^{\infty}\Psi_n$. Now let

$E_1,E_2,\ldots,E_m\varepsilon\bigcup_{n=0}^{\infty}\Psi_n$, then there exists N such that $E_1,\ldots,E_m\varepsilon\Psi_n$

which implies that $\bigcup_{i=1}^{m}E_i = \bigcup_{i=1}^{m}(E_i-\phi)\varepsilon\Psi_{N+1}\subseteq\bigcup_{n=0}^{\infty}\Psi_n$. Thus $\bigcup_{n=0}^{\infty}\Psi_n$ is a field.

Q. E. D.

3.15 <u>Definition</u>. Call a collection of sets, Π, <u>monotone</u> if for all sequences $\{E_n\}\subseteq\Pi$ $\lim_{n\to\infty} E_n\varepsilon\Pi$.

3.16 <u>Lemma</u>. A monotone field is a σ-field and the smallest monotone collection of sets containing a field is a σ-field.

<u>Proof</u>: Let Ω be a monotone field and $\{E_n\}\subseteq\Omega$. Then $\{\bigcup_{i=1}^{n}E_i\}_{n=1}^{\infty}$ is a monotone collection of sets and hence $\lim_{n\to\infty}\bigcup_{i=1}^{n}E_i = \bigcup_{i=1}^{\infty}E_i\varepsilon\Omega$ and Ω is a σ-field.

Now let M be the smallest monotone collection of sets containing a field Ω. I will show that M is a field and hence a σ-field. Denote by [F] the collection of all sets E such that E-F, F-E and $E\cup F\varepsilon M$. Note that $E\varepsilon[F]$ if and only if $F\varepsilon[E]$ and that [E] is non-empty if $E\varepsilon\Omega$. I first show that [E] is monotone. Let $\{E_n\}\subseteq[E]$ be a montone sequence. Then $\lim_{n\to\infty}E_n - F = \lim_{n\to\infty}(E_n-F)\varepsilon M$, $\lim_{n\to\infty}E_n\cup F=\lim_{n\to\infty}(E_n\cup F)\varepsilon M$ and $F-\lim_{n\to\infty}E_n = \lim_{n\to\infty}(F-E_n)\varepsilon M$. Hence $\lim_{n\to\infty}E_n\varepsilon M$.

But for $E \epsilon \Omega$, $F \subseteq [E]$ for all $F \epsilon \Omega$ and since M is minimal, $M \subseteq [E]$ for all $E \epsilon \Omega$. Thus we conclude that if $E \epsilon M$ then $E \epsilon [F]$ for all $F \epsilon \Omega$. Hence $F \epsilon [E]$ for all $F \epsilon \Omega$. Again we may to apply the minimality and we get $M \subseteq [E]$ for all $E \epsilon M$. In particular, this means M satisfies the axioms of a field. Therefore, by the first part of the lemma, M is a σ-field.

Q. E. D.

In particular, I conclude from the lemma that a monotone collection of sets which contains a field contains the σ-field generated by the field.

3.]7 Lemma. Let X and Y be Banach spaces and let $\{T_n\}$ be a sequence of linear maps from X to Y with the following properties:

1.) $\sup_{1 \leq n \leq \infty} |T_n x| < \infty$ for all $x \epsilon X$.

2.) $\lim_{n \to \infty} T_n x$ exists for all x in a fundamental set.

Then $\lim_{n \to \infty} T_n x$ exists for all $x \epsilon X$.

Proof: We note that linear span of a fundamental subset of X is dense in X Therefore, since the T_n are linear, $\lim_{n \to \infty} T_n x$ exists for all x in the linear span of a fundamental set. Therefore, let D be the linear span of the fundamental set. But $\{T_n x\}$ is a pointwise bounded family and by the uniform boundedness principle, for $\epsilon > 0$, there exists δ such that $||x|| < \delta \Rightarrow |T_n x| < \epsilon/3$ for all $x \epsilon D$ and for all n. Thus for $x \epsilon X$, there exists $y \epsilon D$ such that $||x-y|| < \delta$ and there exists N such that for n,p>N we have $|T_n x - T_n y| < \epsilon/3$ and thus

$$|T_n x - T_p x| \leq |T_n x - T_n y| + |T_n y - T_p y| + |T_p y - T_p x|$$
$$< \varepsilon \qquad \text{for } n,p > N.$$

Thus $\{T_n x\}$ is a Cauchy sequence for all $x \varepsilon X$. But Y is Banach and therefore $\lim\limits_{n\to\infty} T_n x$ exists for all $x \varepsilon X$.

Q. E. D.

We are ready to prove the converse of Theorem 3.10 and, because of lemma 3.13, this will provide us with the converse of Theorem 3.11.

3.18 Theorem. Let $K \subseteq Ca(\Sigma)$ have the following properties:

1.) K is bounded.

2.) If $\{E_i\}$ Σ is a sequence decreasing to the void set, then $\lim\limits_{i\to\infty} \lambda(E_i) = 0$ uniformly for $\lambda \varepsilon K$.

Then K is weakly conditionally compact.

Proof: We shall show that K is weakly sequentially compact and apply Theorem 3.9 to show that K is weakly conditionally compact. To this end, let $\{\mu_n\}$ be an arbitrary sequence in K. I will show that $\{\mu_n\}$ has a weakly convergent subsequence.

Define the positive measure $\nu \varepsilon Ca(\Sigma)$ by

$$\nu(E) = \sum_{n=1}^{\infty} 2^{-n} |\mu_n|(E)/[1+|\mu_n|]$$

Let $Ca(\Sigma;\nu) = \{\lambda \varepsilon Ca(\Sigma): \lim\limits_{\nu(E)\to 0} \lambda(E) = 0\}$. First, I will show that $Ca(\Sigma;\nu)$ is a closed linear subspace of $Ca(\Sigma)$. If $\lambda, \lambda' \varepsilon Ca(\Sigma;\nu)$, then clearly $a\lambda + b\lambda'$ is ν-continuous and hence in $Ca(\Sigma;\nu)$. Thus $Ca(\Sigma;\nu)$ is a linear subspace. To see that $Ca(\Sigma;\nu)$ is closed, let $\{\lambda_i\}$ be a sequence in $Ca(\Sigma;\nu)$ such that $\lambda_i \to \lambda$ and let $\varepsilon > 0$ be given. Choose N such that for

$n>N$, $|\lambda-\lambda_n|<\varepsilon/2$ and thus $|(\lambda-\lambda_n)(E)|<\varepsilon/2$ for all E in Σ. By Theorem 2.4, $\lim_{\nu(E)\to 0} \lambda_i(E) = 0$ uniformly for $i=1,2,\ldots$ Hence, choose δ such that $\nu(E)<\delta \Rightarrow |\lambda_i(E)|<\varepsilon/2$ for all i and we have $|\lambda(E)| \leq |\lambda(E)-\lambda_i(E)| + |\lambda_i(E)| <\varepsilon$. Thus, since $\lim_{\nu(E)\to 0} \lambda(E) = 0$, $\lambda\varepsilon Ca(\Sigma;\nu)$ and we conclude that $Ca(\Sigma;\nu)$ is closed.

By the Radon-Nikodym Theorem, there exists an isometric isomorphism between $Ca(\Sigma;\nu)$ and $L(S,\Sigma,\nu)$, the space of ν-integrable functions on (S,Σ) given by

$$\lambda(E) = \int_E f(s)\, d\nu(s) \qquad \text{for all } E\varepsilon\Sigma.$$

Now we choose the sequence $\{f_n\}\subseteq L(S,\Sigma,\nu)$ such that $\lambda_n(E) = \int_E f_n(s)\, d\nu(s)$. Let $\{G_n: n=1,2,\ldots\}$ be a base for the open sets in the scalars and set $E_{m,n} = f_m^{-1}(G_n)$. If we let Σ_1 be the σ-field generated by the $E_{m,n}$, the f_m are measurable with respect to Σ_1 and are therefore in $L(S,\Sigma_1,\nu)$. Setting Σ_2 equal to the field generated by the $E_{m,n}$, we know from lemma 3.14 that Σ_2 is countable. Since Σ_2 is countable, we may use a diagonal process to choose a subsequence $\{\lambda_{n_k}\}$ of $\{\lambda_i\}$ such that $\lim_{k\to\infty} \lambda_{m_k}(E)$ exists for all $E\varepsilon\Sigma_2$. Now let Σ_3 be the collection of all subsets E of Σ_1 such that $\lim_{k\to\infty} \lambda_{m_k}(E)$ exists. Then $\Sigma_2\subseteq\Sigma_3\subseteq\Sigma_1$. I will show that Σ_3 is monotone and by lemma 3.16 we will have $\Sigma_3 = \Sigma_1$. Without loss of generality, let $\{E_i\}\subseteq\Sigma_3$ be a sequence decreasing monotonically to E. Then $\{E_i-E\}$ is a sequence decreasing to the void set. Thus, by the second condition of the hypothesis,

$$\underset{i\to\infty}{\text{limit}}\ \lambda_{m_k}(E_i - E) = 0 \text{ uniformly for } k=1,2,\ldots$$

Hence $\lambda_{m_k}(E) = \underset{i\to\infty}{\text{limit}}\ \lambda_{m_k}(E_i)$ uniformly for $k=1,2,\ldots$

But, $\underset{k\to\infty}{\text{limit}}\ \lambda_{m_k}(E_i)$ exists because $E_i \varepsilon \Sigma_3$ and we conclude that $\underset{k\to\infty}{\text{limit}}\ \lambda_{m_k}(E)$ exists. For, let $\varepsilon > 0$ be given and choose N such that $|\lambda_{m_k}(E_N) - \lambda_{m_k}(E)| < \varepsilon/3$ for $k=1,2,\ldots$ Choose M such that for $j,k > M$ $|\lambda_{m_j}(E_N) - \lambda_{m_k}(E_N)| < \varepsilon/3$. Thus

$$\begin{aligned}|\lambda_{m_j}(E) - \lambda_{m_k}(E)| &\leq |\lambda_{m_j}(E) - \lambda_{m_j}(E_N)| + |\lambda_{m_j}(E_N) - \lambda_{m_k}(E_N)| \\ &\quad + |\lambda_{m_k}(E_N) - \lambda_{m_k}(E)| \\ &< \varepsilon \qquad \text{for } j,k > M.\end{aligned}$$

Thus $\{\lambda_{m_k}(E)\}$ is a Cauchy sequence and $\underset{k\to\infty}{\text{limit}}\ \lambda_{m_k}(E)$ exists and $E \varepsilon \Sigma_3$. As a result, Σ_3 is monotone.

Now, if we wish to show that $\{\lambda_{m_k}\}$ is weakly convergent in $Ca(\Sigma)$, we need only show that $\{\lambda_{m_k}\}$ is convergent in the closed subspace $Ca(\Sigma;\nu)$. For suppose $\lambda_{m_k} \to \lambda$ weakly in $Ca(\Sigma)$ and let f be an arbitrary continuous linear functional on $Ca(\Sigma)$. If $f_0 = f|_{Ca(\Sigma;\nu)}$, then $f_0(\lambda_{m_k}) \to f_0(\lambda)$ implies that $f(\lambda_{m_k}) \to f(\lambda)$. Thus $\lambda_{m_k} \to \lambda$ weakly in $Ca(\Sigma;\nu)$. However, because of the correspondence between the λ_{m_k} and the f_{m_k}, the problem is reduced to showing that $\{f_{m_k}\}$ is weakly convergent in $L(S,\Sigma_1,\nu)$. But we note that $\{\lambda_{m_k}\} \subseteq K$ which is bounded by the first condition of the hypothesis and therefore, $\{f_{m_k}\}$

is bounded. Thus we may apply lemma 3.17 and $\{f_{m_k}\}$ will be convergent on $L(S, \Sigma_1, \nu)$ if they are shown to be convergent on a fundamental set of continuous linear functionals on $L(S, \Sigma_1, \nu)$. We choose as the fundamental set $D = \{g_E: E \varepsilon \Sigma_1\}$ where we define

$g_E = \int_S \lambda_E f(s) d\nu(s)$. But note that

$$\underset{k\to\infty}{\text{limit}}\ g_E(f_{m_k}) = \underset{k\to\infty}{\text{limit}} \int_S \chi_E f_{m_k}(s) d\nu(s)$$

$$= \underset{k\to\infty}{\text{limit}} \int_E f_{m_k}(s) d\nu(s)$$

$$= \underset{k\to\infty}{\text{limit}}\ \lambda_{m_k}(E)$$ and this limit has already been show to exist for all $E \varepsilon \Sigma$. Thus $\underset{k\to\infty}{\text{limit}}\ g_E(f_{m_k})$ exists for all g_E in D.

The theorem's proof will be concluded by showing that D is indeed a fundamental set. By the Riesz Representation Theorem, for $x^* \varepsilon L^*$, there exists $g \varepsilon L^\infty$ such that

$x^*(f) = \int_S f(s) g(s) d\nu(s)$, where the norm for g in L^∞ is given by

$$||g|| = \text{ess-sup}|g(t)| = \inf\{M: m\{t: g(t) > M\} = 0\} < \infty.$$

~~Thus it is sufficient to show that~~ χ_E is fundamental over L^∞. Let $g \varepsilon L$ and let $M = \text{ess-sup}\ |g(t)| + 1$. For $\varepsilon > 0$, choose n such that $M/n < \varepsilon$ and let

$$E_k = \{x: kM/n \leq g(x) \leq (k+1)M/n\} \quad k=-n, -n+1, \ldots, n-1.$$

Set $f = \sum_{k=-n}^{k=n-1} k\chi_{E_k}$. Thus, we have for all x in $g^{-1}(-M,M)$

$|g(s)-f(x)| \leq M/n$ and $m\{x: x \notin g^{-1}(-M,M)\} = 0$ by the definition of M. Thus ess-sup $|(f-g)(t)| \leq M/n < \varepsilon$ and the set $\{\chi_E\}$ is fundamental.

Q. E. D.

Chapter IV

INTEGRATION

In this chapter, let X denote a real or complex Banach Space and X* its conjugate. As usual, Σ will denote a σ-field of subsets of a set S. We will develop a theory for integration of scalar functions with respect to a vector measure and the resulting will take values in the Banach Space X. This integral will then be shown to satisfy the usual properties of integrals.

4.1 Definition. The additive set function $\mu: \Sigma \to X$ is called a vector measure if $x^*\mu \in Ca(\Sigma)$ for all $x^* \in X^*$.

4.2 Definition. Given the vector measure μ, call the set function $||\mu||$ the semivariation of μ where

$||\mu||(E) = \text{Sup } |\sum_{i=1}^{n} \alpha_i \mu(E_i)|$ for $E \in \Sigma$, the supremum taken over all scalars α_i such that $|\alpha_i| \leq 1$ and disjoint partitions of E into measurable sets.

4.3 Lemma. 1.) If $A \subseteq B$, then $||\mu||(A) \leq ||\mu||(B)$

2.) $||\mu||(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} ||\mu||(E_i)$ for all sequences $\{E_i\} \subseteq \Sigma$.

Proof: 1.) Let $E_1, E_2, \ldots, E_n \in \Sigma$ such that the E_i are disjoint and $A = \bigcup_{i=1}^{n} E_i$. Then

$|\sum_{i=1}^{n} \alpha_i \mu(E_i)| = |\sum_{i=1}^{n} \alpha_i \mu(E_i) + 0\mu(B-A)| \leq ||\mu||(B)$ and we

we have that $||\mu||(A) \leq ||\mu||(B)$.

2.) Let $\{E_i\} \subseteq \Sigma$ and let $F_n = \bigcup_{i=1}^{n} E_i$. Now if $E = \bigcup_{i=1}^{\infty} E_i$, we can assume that the E_i are disjoint. Now let $F_n = \bigcup_{i=1}^{m} S_i$ be a partition of F_n into disjoint measurable sets.

Then
$$\begin{aligned} \left|\sum_{i=1}^{m} \alpha_i \mu(S_i)\right| &= \left|\sum_{i=1}^{m} \alpha_i \mu\left(S_i \cap \bigcup_{j=1}^{n} E_j\right)\right| \\ &= \left|\sum_{i=1}^{m} \alpha_i \sum_{j=1}^{n} \mu(S_i \cap E_j)\right| \\ &\leqq \sum_{j=1}^{n} \left|\sum_{i=1}^{m} \alpha_i \mu(S_i \cap E_j)\right| \\ &\leqq \sum_{j=1}^{n} ||\mu||(E_j) \\ &\leqq \sum_{j=1}^{\infty} ||\mu||(E_j). \end{aligned}$$

Thus we conclude that $||\mu||\left(\bigcup_{i=1}^{n} E_i\right) \leqq \sum_{j=1}^{\infty} ||\mu||(E_j)$ and thus $||\mu||\left(\bigcup_{i=1}^{\infty} E_i\right) \leqq \sum_{j=1}^{\infty} ||\mu||(E_j)$.

Q. E. D.

4.4 <u>Lemma</u>. 1.) If $\mu: \Sigma \to X$ is vector measure and $E_1, E_2, \ldots, E_n$ is a disjoint sequence in Σ, then

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i).$$

2.) $||\mu||(S) < \infty$ and for $E \in \Sigma$,

$$||\mu||(E) \leqq 4 \text{ Sup } \{|\mu(F)| : F \in \Sigma,\ F\ E\} < \infty.$$

<u>Proof</u>: 1.) Let $x^* \in X^*$. Noting that $\mu\left(\bigcup_{i=1}^{\infty} E_i\right)$ and $\sum_{i=1}^{\infty} \mu(E_i)$ are both finite, we have that

$$x^*\left[\mu\left(\bigcup_{i=1}^{\infty} E_i\right) - \sum_{i=1}^{\infty} \mu(E_i)\right] = x^*\mu\left(\bigcup_{i=1}^{\infty} E_i\right) - x^* \sum_{i=1}^{\infty} \mu(E_i) = \sum_{i=1}^{\infty} x^*\mu(E_i - E_i) = 0.$$

However, X^* is total over X and thus

$$\mu(\bigcup_{i=1}^{\infty} E_i) - \sum_{i=1}^{\infty} \mu(E_i) = 0.$$

2.) $\text{Sup}\,\{|x^*\mu(F)|: F\varepsilon\Sigma,\ F\subseteq E\}$ is finite for $x^*\varepsilon X^*$ by the definition of μ. Then, by Corollary 3.6, $\text{Sup}\,\{|\mu(F)|\}$ is bounded. However, as in Lemma 3.3, $v(x^*\mu,E) \leqq 4 \sup_{F\ E} |x^*\mu(F)|$ and as a result

$$||\mu||(E) = \text{Sup}\ |\sum_{i=1}^{n} \alpha_i \mu(E_i)|$$

$$= \sup_{|x^*|\leqq 1} \text{Sup}\ |\sum_{i=1}^{n} \alpha_i x^*\mu(E_i)|$$

$$\leqq \sup_{|x^*|\leqq 1} \text{Sup} \sum_{i=1}^{n} |\alpha_i| v(x^*\mu, E_i)$$

$$\leqq \sup_{|x^*|\leqq 1} v(x^*\mu, E)$$

$$\leqq 4 \sup_{|x^*|\leqq 1} \sup_{F\subseteq E} |x^*\mu(F)|$$

$$\leqq 4 \sup_{F\subseteq E} |\mu(F)|$$ since, by the Hahn-Banach Theorem, $\sup_{|x^*|\leqq 1} |x^*s| = |s|$.

Q. E. D.

The following lemma and its corollary will prove quite useful in the proving of standard properties for our yet to be defined integral.

4.5 <u>Lemma</u>. $D = \{x^*\mu: x^*\varepsilon X^*,\ |x^*|\leqq 1\}$ is weakly conditionally compact.

<u>Proof</u>: We shall prove the lemma by showing that D satisfies the hypotheses of Theorem 3.18. It is clear that D is bounded. Thus we need only show that D satisfies the second hypothesis

of the theorem. Let $\{E_i\} \subseteq \Sigma$ be a sequence decreasing to the void set. Since μ is countably additive,

$$\lim_{n\to\infty} \mu(E_n) = \mu\left(\bigcap_{i=1}^{\infty} E_n\right) = \mu(\phi) = 0$$

Hence, $|x^*\mu(E_i)| \leqq |x^*||\mu(E_i)| \leqq |\mu(E_i)|$ for all $x^*\mu\epsilon D$ and $\lim_{n\to\infty} x^*\mu(E_n) = 0$ uniformly for elements of D. Thus the hypotheses of Theorem 3.18 are satisfied and the result follows.

Q. E. D.

4.6 <u>Corollary</u>. If μ is vector measure, there exists positive measure $\nu\epsilon Ca(\Sigma)$ such that

1.) $\lim_{\nu(E)\to 0} \mu(E) = 0$

2.) $\lim_{\nu(E)\to 0} ||\mu||(E) = 0.$

3.) (ν may be so chosen that) $\nu(E) < \varepsilon$ whenever $||\mu||(E) < \varepsilon$.

<u>Proof</u>: 1.) Let $D = \{x^*\mu: x^*\epsilon X^*, |x^*| \leqq 1\}$. Since D is weakly conditionally compact, by Theorem 3.11 there exists a positive $\nu\epsilon Ca(\Sigma)$ such that $\lim_{\nu(E)\to 0} x^*\mu = 0$ uniformly for $x^*\mu\epsilon D$.

Let $\varepsilon > 0$ be given and choose δ such that $\nu(E) < \delta$ implies that $|x^*\mu(E)| < \varepsilon$ for all $x^*\mu\epsilon D$. Then, by the Hahn-Banach Theorem, $|\mu(E)| = \sup_{|x^*|\leqq 1} |x^*\mu| \leq \varepsilon$ and $\lim_{\nu(E)\to 0} |\mu(E)| = 0$.

2.) Let $\varepsilon > 0$ be given and choose $\delta > 0$ such that $\nu(E) < \delta$ implies that $|\mu(E)| < \varepsilon/4$. Now if $F \subseteq E$, then $\nu(F) < \nu(E) < \delta$ and thus $|\mu(F)| < \varepsilon/4$.

Thus, by Lemma 4.4,

$$||\mu||(E) \leq 4 \text{ Sup } \{|\mu(F)|: F \varepsilon \Sigma, F \quad E\}$$

$$< 4(\varepsilon/4)$$

$$\leq \varepsilon$$

and we have that $\lim\limits_{\nu(E)\to 0} ||\mu||(E) = 0$.

3.) By Corollary 3.12, ν may be chosen such that $|x^*\mu(E)| < \varepsilon$ for all $x^*\mu \varepsilon D$ if and only if $|\mu(E)| < \varepsilon$ and the conclusion follows.

Q. E. D.

Now we are in a position to develop a theory of integration of scalar functions with respect to the vector measure μ.

4.7 Definition. Call a set E μ-null if $||\mu||(E) = 0$; a property is said to hold μ-almost everywhere, abbreviated μ-a.e., if it holds on the complement of a μ-null set.

4.8 Lemma. If $E_i \varepsilon \Sigma, (i=1,2,\dots,n)$ are μ-null, then $\bigcup\limits_{i=1}^{\infty} E_i$ is μ-null.

Proof: $||\mu||(\bigcup\limits_{i=1}^{\infty} E_i) \leq \sum\limits_{i=1}^{\infty} ||\mu||(E_i) = 0$ by Lemma 4.3.

Q. E. D.

4.9 Definition. Let f be a scalar function defined on S. Call f measurable if for every Borel set of scalars, B, $f^{-1}(B) \varepsilon \Sigma$.

4.10 Definition. A finite linear combination of characteristic functions of measurable sets is called a simple function. If $\phi = \sum\limits_{i=1}^{n} \alpha_i \chi_{E_i}$, let $\{a_1, a_2, \dots, a_m\}$ be the distinct values that ϕ takes on. Let $A_i = \{x: \phi(x) = \alpha_i\}$ for $i=1,2,\ldots,m$. Then ($A_i \cap A_j = \phi$ for $i \neq j$ and) call

$$\phi = \sum_{j=1}^{m} \alpha_j \chi_{A_j} \quad \text{the } \underline{\text{canonical}}\ \underline{\text{representation}} \text{ of } \phi.$$

4.11 Definition. Given the simple function ϕ with canonical representation $\phi = \sum_{j=1}^{m} a_j \chi_{A_j}$, define the integral of ϕ over S by $\int \phi(s)d\mu(s) = \sum_{j=1}^{m} a_j \mu(A_j)$ and for $E \in \Sigma$ define the integral of ϕ over E by $\int_E \phi(s)d\mu(s) = \int \phi(s)\chi_E d\mu(s) = \sum_{j=1}^{m} a_j \mu(A_j \cap E)$.

4.12 Lemma. If $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ where $E_i \cap E_j = \phi$, then

$$\int \phi(s)d\mu(s) = \sum_{i=1}^{n} a_i \mu(E_i).$$

Proof: $A_a = \{x: \phi(x)=a\} = \bigcup_{a_i=a} E_i$ and thus $a\mu(A_i) = \sum_{a_i=a} a_i \mu(E_i)$.

Hence $\int \phi(s)d\mu(s) = \sum a\mu(A_a) = \sum_{i=1}^{n} a_i \mu(E_i)$.

Q. E. D.

4.13 Lemma. If f and g are simple functions, then

$$\int (af + bg) = a\int f + b\int g.$$

Proof: Let f and g have the canonical representations

$f = \sum_{j=1}^{n} a_j \chi_{A_j}$ and $g = \sum_{i=1}^{m} b_i \chi_{B_i}$. Then, if $E_{ji} = A_j \cap B_i$, we may write $f = \sum_{j=1}^{n} \sum_{i=1}^{m} a_j \chi_{E_{ji}}$ and $g = \sum_{i=1}^{m} \sum_{j=1}^{n} b_i \chi_{E_{ji}}$.

$af + bg = \sum_{j=1}^{n} \sum_{i=1}^{m} (aa_j + bb_i)\mu(E_{ji})$ and by lemma 4.12

$$\int af + bg = \sum_{j=1}^{n} \sum_{i=1}^{m} (aa_j + bb_i)\mu(E_{ji})$$

$$= a\sum_{j=1}^{n} a_j \sum_{i=1}^{m} \mu(E_{ji}) + b\sum_{i=1}^{m} b_i \sum_{j=1}^{n} \mu(E_{ji})$$

$$= a\int f + b\int g.$$

Q. E. D.

In light of these two lemmas, for integration it is neither necessary that a simple function be in canonical form nor necessary that it be composed of characteristic functions defined disjoint sets. Thus if

$$f = \sum_{i=1}^{n} a_i \chi_{E_i}, \int f = \sum_{i=1}^{n} a_i \mu(E_i)$$ regardless of the form of the representation of f.

4.14 <u>Lemma</u>. Let f be a simple function. Then $\int f$ is a countably additive set function in X.

<u>Proof</u>: Let $\{E_i\} \subseteq \Sigma$ be a disjoint sequence and let $f = \sum_{i=1}^{n} a_i \chi_{A_i}$.

Lemma 4.4 yields:
$$\int_{\cup E_i} f = \sum_{i=1}^{n} a_i \mu(A_i \cap \bigcup_{j=1}^{\infty} E_j)$$

$$= \sum_{i=1}^{n} a_i \sum_{j=1}^{\infty} \mu(A_i \cap E_j)$$

$$= \sum_{j=1}^{\infty} \sum_{i=1}^{n} a_i \mu(A_i \cap E_j)$$

$$= \sum_{j=1}^{\infty} \int_{E_j} f$$

Q. E. D.

We make the observation that given $E \in \Sigma$ and given the simple function

$$f = \sum_{i=1}^{n} a_i \chi_{E_i},$$ we have the following:

$$\left|\int_E f(s)\,d\mu(s)\right| = \left|\sum_{i=1}^{n} a_i \mu(E_i\ E)\right|$$

$$\leq \operatorname*{Sup}_{s\in E} |f(s)|\ \{\operatorname{Sup} \left|\sum_{i=1}^{n} \alpha_i\ \mu(E_i\ E)\right|\}$$

$$\leq \operatorname*{Sup}_{s\in E} |f(s)|\cdot||\mu||(E)$$

where $\alpha_i = a_i/[\operatorname{Sup}|f(s)|]$ and thus $|\alpha_i| \leq 1$.

Now we would like to extend the definition of an integral to a more general class of functions.

4.15 Definition. The measurable function f is called μ-integrable if there exists a sequence $\{f_n\}$ of simple functions with the following properties:

1.) $f_n \to f$ μ-a. e.

2.) $\{\int_E f_n\}$ converges in the norm of X for all $E\varepsilon\Sigma$.

For $E\varepsilon\Sigma$, call $\int_E f(s)d\mu(s) = \lim_{n\to\infty} \int_E f_n(s)d\mu(s)$ the integral of f with respect to μ over the set E.

With the following definition, we will have a sufficient condition for integrability.

4.16 Definition. Let f be a measurable function. Then call $\operatorname*{ess\text{-}sup}_{s\varepsilon E} |f(s)| = \operatorname{Inf}\{M: \mu\{s\varepsilon E: |f(s)| > M\} = 0\}$ the μ-essential supremum of f on E. f is called μ-essentially bounded if $\operatorname*{ess\text{-}sup}_{s\varepsilon E} |f(s)| < \infty$.

4.17 Theorem. If $E\varepsilon\Sigma$ and f is μ-integrable, then $\int_E f(s)d\mu(s)$ is well defined.

Proof: Let $\{f_n\}$ and $\{g_n\}$ be two sequences of simple functions such that $f_n \to f$ and $g_n \to f$ except on the μ-null set A and such that $\{\int_E f_n\}$ and $\{\int_E g_n\}$ converge for all $E\in\Sigma$. Define the simple function h_n by

$$h_n(s) = \begin{cases} f_n(s)-g_n(s) & \text{for } s\in S-A \\ 0 & \text{for } s\in A \end{cases}$$

Then $h_n(s) \to 0$ for all $s\in S$. It must be shown that $\{\int_E h_n\}$ converges to 0 in X for $E\in\Sigma$.

By Corollary 4.6 there exists positive $\nu\in Ca(\Sigma)$ such that $\lim\limits_{\nu(E)\to 0} ||\mu||(E) = 0$. Then for fixed n

$$\begin{aligned}\lim_{\nu(E)\to 0} \left|\int_E h_n(s)d\mu(s)\right| &\leqq \lim_{\nu(E)\to 0} \{\sup_{s\in E} |h_n(s)|\cdot||\mu||(E)\} \\ &\leqq \sup_{s\in S} |h_n(s)| \lim_{\nu(E)\to 0} ||\mu||(E) \\ &= 0\end{aligned}$$

But $\{\int_E h_n\}$ converges for all $E\in\Sigma$ since it is the difference of two convergent sequences. Thus we may apply Theorem 2.4. For $\varepsilon>0$ there exists δ such that $\nu(D)<\delta$ implies that $|\int_D h_n| < \varepsilon/2$ for all n. Now, by Egoroff's Theorem, choose $D\in\Sigma$ such that $\nu(D)<\delta$ and $h_n \to 0$ uniformly on S-D. Therefore choose N such that n>N implies that $|h_n| < \varepsilon/[2\cdot||\mu||(S)]$ on S-D and thus n>N implies that

$$\begin{aligned}\left|\int_E h_n\right| &\leqq \left|\int_{E-D} h_n\right| + \left|\int_{D\cap E} h_n\right| \\ &< ||\mu||(E-D)\varepsilon/[2\cdot||\mu||(S)] + \varepsilon/2 \\ &< \varepsilon\end{aligned}$$

Q. E. D.

4.18 Theorem. 1.) The set of integrable functions is a linear space and for $E\in\Sigma$ $\int_E f$ is a linear map of this space into X. 2.) If f is a measurable function which is μ-essentially bounded on S, then f is μ-integrable and

$$\left|\int f\right| \leqq \{\mu\text{-ess-sup}_{s\in E} |f(s)|\}\cdot\|\mu\|(S).$$

Proof: 1.) Let $\Phi = \{f: f \text{ is } \mu\text{-integrable}\}$. If $f,g\in\Phi$, then let $\{f_n\}$, $\{g_n\}$ be sequences of simple functions converging μ-a.e. to f and g respectively such that $\{\int_E f_n\}$ and $\{\int_E g_n\}$ converge for all $E\in\Sigma$. Thus $\{\int_E af_n\}$ and $\{\int_E f_n+g_n\}$ converge for all $E\in\Sigma$. However, $af_n \to af$ and $f_n+g_n \to f+g$ μ-a.e. and af and f+g are μ-integrable. The facts that $\int_E af = a\int_E f$, $\int_E (f+g) = \int_E f + \int_E g$, and $\{\int_E g_n\}$ and $\{\int_E (f_n+g_n)\}$ converge all follow from the equations:

$$\int_E af = \operatorname*{limit}_{n\to\infty} \int_E af_n = a\cdot\operatorname*{limit}_{n\to\infty} \int_E f_n = a\int_E f$$

and $$\int_E f+g = \operatorname*{limit}_{n\to\infty} \int_E f_n+g_n = \operatorname*{limit}_{n\to\infty} \int_E f_n + \int_E g_n = \int_E f + \int_E g$$

2.) For $\varepsilon>0$, let $M = \operatorname*{ess-sup}_{s\in S} |f(s)| + \varepsilon$.

For $n=2^m$, $m=1,2,\ldots$ define $A_k^m = \{x: M(k-1)/n \leqq f(x) \leqq Mk/n\}$ for $k=-n+1,-n+2,\ldots,n$. Let $B_m = S-\left(\bigcup_{-n+1}^{n} A_k^m\right)$ and thus

$\|\mu\|(B_m) = 0$ by the way M was chosen. If we define

$f_n = M/n \sum_{-n+1}^{n} k\chi_{A_k}$, then $|f_n-f| < M/n$ except on B_m.

Thus $f_n \to f$ except possibly on $\bigcup_{m=1}^{\infty} B_m$. But, by lemma 4.8, $\|\mu\|\left(\bigcup_{m=1}^{\infty} B_m\right) = 0$ and we have that $f_n \to f$ μ-a.e.

Without loss of generality, let p be greater than q and we have $|\int_E f_p - \int_E f_q| = |\int_E (f_p - f_q)| \leqq \frac{M}{q}|\mu|(E)$.

Thus $\{\int_E f_n\}$ is a Cauchy sequence and as a result converges for $E \varepsilon \Sigma$. Thus f is μ-integrable by definition.

Finally, by our choice of the f_n

$$\begin{aligned} |\int f(s)d\mu(s)| &= \lim_{n\to\infty} |\int f_n(s)d\mu(s)| \\ &\leqq \lim_{n\to\infty} \{\sup_{s\varepsilon S} |f_n(s)|\cdot||\mu||(S)\} \\ &\leqq M\cdot||\mu||(S) \\ &\leqq \{\operatorname{ess-sup}_{s\varepsilon S} |f(s)|+\varepsilon\}||\mu||(S). \end{aligned}$$

Thus $|\int f(s)d\mu(s)| \leqq \{\operatorname{ess-sup}_{s\varepsilon S} |f(s)|\}\cdot||\mu||(S)$.

Q. E. D.

4.19 <u>Theorem</u>. Let D be a Banach Space and let $U: X \to D$ be a bounded linear operator. Then

1.) $U\mu$ is vector measure from Σ to D.

2.) For any f which is μ-integrable and $E \varepsilon \Sigma$

$$U\{\int_E f(s)d\mu(s)\} = \int_E f(s)dU\mu(s).$$

<u>Proof</u>: 1.) We must show that $y^*(U\mu)\varepsilon Ca(\Sigma)$ for all $y^*\varepsilon D^*$. But for $y^*\varepsilon D^*$, $y^*\circ U$ is a continuous linear functional on X which means that $y^*\circ U\varepsilon X^*$ and hence $(y^*\circ U)\mu\varepsilon Ca(\Sigma)$. Thus $y^*(U\mu)\varepsilon Ca(\Sigma)$ for all $y^*\varepsilon D^*$.

2.) First let $f = \sum_{i=1}^{n} a_i\chi_{E_i}$ be a simple function.

Then $U\{\int_E f\} = U\{\sum_{i=1}^{n} a_i\mu(E_i)\} = \sum_{i=1}^{n} a_iU\mu(E_i) = \int_E f(s)dU\mu(s)$ and the result holds for simple functions. Now let f be μ-integrable and let $\{f_n\}$ be a sequence of simple functions such that $f_n \to f$ μ-a.e. and such that $\{\int_E f_n\}$ converges for all $E \varepsilon \Sigma$. Then, since U is a continuous, linear operator,

$$U(\int_E f(s)d\mu(s)) = U(\lim_{n\to\infty} \int_E f_n(s)d\mu(s))$$

$$= \lim_{n\to\infty} \int_E f_n(s)dU\mu(s)$$

$$= \int_E f(s)dU\mu(s).$$

Q. E. D.

4.20 Theorem. Let f be μ-integrable. Then

$$\lim_{||\mu||(E)\to 0} \int_E f(s)d\mu(s) = 0.$$

Proof: Let $\{f_n\}$ be a sequence of simple functions such that $f_n \to f$ μ-a.e. and such that $\{\int_E f_n\}$ is convergent for all $E \varepsilon \Sigma$. By Corollary 4.6, I may choose positive measure $\nu \varepsilon Ca(\Sigma)$ such that $\lim_{\nu(E)\to 0} ||\mu||(E) = 0$. Then, as in Theorem 4.17,

$$\lim_{\nu(E)\to 0} \int_E f_n = 0 \qquad \text{for } n=1,2,\ldots$$

Thus, by Theorem 2.4, this limit is uniform in n. Since ν may be chosen such that $\nu(E) < \varepsilon$ whenever $||\mu||(E) < \varepsilon$, $\lim_{||\mu||(E)\to 0} \int_E f_n = 0$ uniformly in n and thus

$$\lim_{||\mu||(E)\to 0} \int_E f = 0.$$

Q. E. D.

4.21 Theorem. Let f be μ-integrable. Then $\int_E f$ is a countably additive set function on Σ to X.

Proof: Let $\{E_i\}$ be a disjoint sequence of sets in Σ. Then if $x^* \varepsilon X^*$, by Theorem 4.19, we have

$$
\begin{aligned}
x^*\left(\int_{\cup E_i} f(s)d\mu(s)\right) &= \int_{\cup E_i} f(s)dx^*\mu(s) \\
&= \sum_{i=1}^{\infty} \int_{E_i} f(s)dx^*\mu(s) \\
&= x^*\left[\sum_{i=1}^{\infty} \int_{E_i} f(s)d\mu(s)\right].
\end{aligned}
$$

However, X^* is total over X and thus

$$\int_{\cup E_i} f(s)d\mu(s) = \sum \int_{E_i} f(s)d\mu(s).$$

Q. E. D.

4.22 Theorem. Let $\{f_n\}$ be a sequence of μ-integrable functions such that $f_n \to f$ μ-a.e. If $\lim_{||\mu||(E)\to 0} \int_E f_n(s)d\mu(s) = 0$ uniformly in n, then f is μ-integrable and

$$\int_E f(s)d\mu(s) = \lim_{n\to\infty} \int_E f_n(s)d\mu(s).$$

Proof: We shall first show that $\{\int_E f_n\}$ converges for all $E \varepsilon \Sigma$. Fix the positive integer k and let δ_k be such that $2^{-k} > \delta_k > 0$ and such that for $E \varepsilon \Sigma$ with $||\mu||(E) < \delta_k$

$$\left|\int_E f_n\right| < 2^{-k} \qquad \text{for } n=1,2,\ldots$$

Choose $\nu \varepsilon Ca(\Sigma)$ as in Corollary 4.6 and let $\eta_k > 0$ be such that for $A \varepsilon \Sigma$ with $\nu(A) < \eta_k$ we have $||\mu||(A) < \delta_k$. By Corollary 4.6, if $||\mu||(A) = 0$, then $\nu(A) = 0$ and hence if $f_n \to f$ μ-a.e., then $f_n \to f$ ν-a.e. By Egoroff's Theorem, there exists $A \varepsilon \Sigma$ such that $\nu(A) < \eta_k$ and $f_n \to f$ uniformly on $S-A$. Choose N_k such that $n,m \geqq N_k$ implies that $|f_n(s) - f_m(s)| < 2^{-k}$ for all $s \varepsilon S-A$. Thus for $E \varepsilon \Sigma$ and $n,m \geqq N_k$ we have

$$(1)\qquad |\int_E (f_n-f_m)| \leq |\int_{E-A} (f_n-f_m)| + |\int_{E\cap A} f_n| + |\int_{E\cap A} f_m| < 2^{-k}[||\mu||(S) + 2].$$

However k was arbitrary and thus $\{\int_E f_n\}$ converges for all $E\varepsilon\Sigma$.

Now we show that f is μ-integrable. For each k, let δ_k and η_k be chosen as above. Since f_k is μ-integrable, there exists a simple function g_k and a set $A_k\varepsilon\Sigma$ such that $\nu(A_k)<\eta_k$ and such that

$$(2)\qquad |f_k-g_k| < 2^{-k} \text{ on } S-A_k \text{ and}$$

$$(3)\qquad |g_k(s)| < 2|f_k(s)| \text{ for } s\varepsilon S.$$

If we set

$$B_k = \bigcup_{i=k}^{\infty} A_i$$

we have that $B_k\varepsilon\Sigma$ and $\{B_k\}$ decreases to the set $B = \bigcap_{i=1}^{\infty} B_i$. Observing that $||\mu||(B_k)\leq \sum_{i=k}^{\infty} ||\mu||(A_i)\leq \sum_{i=k}^{\infty} \delta_i \leq \sum_{i=k}^{\infty} 2^{-i} = 2^{-k+1}$, we see that $||\mu||(B) < 2^{-k}$ for all k and thus $||\mu||(B) = 0$. Now, for $s\varepsilon S-B$

$$|f(s)-g_k(s)| \leq |f(s)-f_k(s)| + |f_k(s)-g_k(s)| < 2^{-k} + \varepsilon$$ for sufficiently large k and $g_k \to f$ μ-a.e.

To show f is μ-integrable we have to show that $\{\int_E g_n\}$ converges for all $E\varepsilon\Sigma$. First observe that for $||\mu||(E)<\delta_k$

$$|\int_E f_n(s)d\mu(s)| < 2^{-k}.$$

Then I claim that

$$\int_E |f_n(s)|\cdot|dx^*\mu(s)| < 16\cdot 2^{-k} \text{ for } |x^*| \leq 1.$$ To see this, let $E = E_1 \cup E_2$ be the Hahn Decomposition of E with respect to the measure λ where for subsets $F\subseteq E$ $\lambda(F) = \int_F f_n(s)\,|dx^*\mu(s)|$.

Similarly, let $E_i = F_1^i \cup F_2^i$ be the Hahn Decomposition of E_i with respect to the measure $\int f_n(s)dx^*\mu(s)$. Then

$$|\int_{F_j^i} f_n(s)dx^*\mu(s)| \leqq 4 \sup_{F\subseteq F_j^i} |\int_F f_n(s)dx^*\mu(s)|$$
$$< 4\cdot 2^{-k}$$

since $F\subseteq F_j^i \subseteq E$ implies that $||\mu||(F) \leqq ||\mu||(E) < \delta_k$. The assertion then follows.

Since $||\mu||(E\cap A_k) < \delta_k$, we have from (3)

$$|\int_{E\cap A_k} g_k(s)d\mu(s)| \leqq \sup_{|x^*|\leqq 1} \int_{E\cap A_k} |g_k(s)|\cdot|dx^*\mu(s)|$$
$$\leqq 2 \sup_{|x^*|\leqq 1} \int_{E\cap A_k} |f_k(s)|\cdot|dx^*\mu(s)|$$
$$\leqq 32\cdot 2^{-k}.$$

This result, along with (2), yields

$$|\int_E (f_k-g_k)| \leqq |\int_{E-A_k} (f_k-g_k)| + |\int_{E\cap A_k} f_k| + |\int_{E\cap A_k} g_k|$$
$$\leqq 2^{-k}[||\mu||(S)] + 2^{-k} + 32\cdot 2^{-k}$$
$$\leqq 2^{-k}[||\mu||(S) + 33].$$

Therefore there exists an N such that n,m>N implies that

$$|\int_E (g_n-g_m)| \leqq |\int_E (g_n-f_n)| + |\int_E (f_n-f_m)| + |\int_E (f_m-g_m)|$$
$$\leqq 2\cdot 2^{-k}[||\mu||(S)+33] + 2^{-k}[||\mu||(S)+2]$$
$$\leqq 2^{-k} M.$$

This shows that the sequence $\{\int_E g_n\}$ is a Cauchy sequence and is therefore convergent.

Finally, we show that $\int_E f = \lim_{n\to\infty} \int_E f_n$. To this end, let $h_n(x) = f(x) - f_n(x)$ for n=1,2,... and it is seen that the h_n are μ-integrable. Then, by Theorem 4.20 we have

$$\lim_{||\mu||(E)\to 0} \int_E h_n = 0 \text{ for } n=1,2,\ldots$$

However, the sequence $\{\int_E h_n\}$ is convergent for all $E\varepsilon\Sigma$ and we may apply Theorem 2.4 to obtain $\lim\limits_{||\mu||(E)\to 0} \int_E h_n = 0$ uniformly in n. Therefore, noting that $h_n \to 0$ μ-a.e., we may apply the first result of this theorem and $\lim\limits_{n\to\infty} \int_E h_n = 0$ from which the result follows.

Q. E. D.

As a final result we present the bounded convergence theorem for vector measures.

4.23 <u>Theorem</u>. Let $\{f_n\}$ be a sequence of μ-integrable functions such that $f_n \to f$ μ-a.e. and let g be μ-integrable such that $|f_n(x)| \leqq g(x)$ for $n=1,2,\ldots$ Then f is μ-integrable and $\int_E f = \lim\limits_{n\to\infty} \int_E f_n$ for all $E\varepsilon\Sigma$.

<u>Proof</u>: In light of Theorem 4.22, we need only show that $\lim\limits_{||\mu||(E)\to 0} \int_E f_n = 0$ uniformly in n. For $\varepsilon>0$, by Theorem 4.20 we may choose $\delta>0$ such that for $E\varepsilon\Sigma$ with $||\mu||(E) < \delta$ we have $|\int_E g| < \varepsilon/4$. Thus, as in the proof of Theorem 4.22,

$\int_E g(s)|dx^*\mu(s)| < \varepsilon$ for $|x^*| \leqq 1$. Thus for $||\mu||(E)<\delta$

$$\begin{aligned} \left|\int_E f_n(s)d\mu(s)\right| &\leqq \sup_{|x^*|\leqq 1} \int_E |f_n(s)|\cdot|dx^*\mu(s)| \\ &\leqq \sup_{|x^*|\leqq 1} \int_E g(s)\ |dx^*\mu(s)| \\ &< \varepsilon \qquad \text{for } n=1,2,\ldots \end{aligned}$$

Q. E. D.

Appendix

In the first part of the appendix we list the proofs of elementary theorems unproved in the paper. We will follow these with the deeper theorems that were used without proof along with references.

Theorem A: A field Σ is closed under differences, symmetric differences and finite intersections.

Proof: Let $A_1, A_2, \ldots, A_n \varepsilon \Sigma$. Then $S-A_i \varepsilon \Sigma$ for $i=1,2,\ldots,n$. Hence $\bigcup_{i=1}^{n}(S-A_i) \varepsilon \Sigma$ which implies that $S-\bigcup_{i=1}^{n}(S-A_i) \varepsilon \Sigma$. But $S-\bigcup_{i=1}^{n}(S-A_i) = \bigcap_{i=1}^{n} A_i$.

Now if $A,B \varepsilon \Sigma$, $S-B \varepsilon \Sigma$ and thus $A-B = A \cap (S-B) \varepsilon \Sigma$.

Finally $(A-B), (B-A) \varepsilon \Sigma$ implies that

$A \Delta B = (A-B) \cup (B-A) \varepsilon \Sigma$.

Q. E. D.

Theorem B: A σ-field is closed under countable intersections.

Proof: Let $A_1, A_2, \ldots \varepsilon \Sigma$ a σ-field. Then $S-A_i \varepsilon \Sigma$ for $i=1,2,\ldots$ Thus

$$\bigcap_{i=1}^{\infty} A_i = S - \bigcup_{i=1}^{\infty}(S-A_i) \varepsilon \Sigma.$$

Q. E. D.

Theorem C (Uniform Boundedness Principle) (Wilansky [9, p. 117]): Let Φ be a pointwise bounded family of continuous semi-norms on complete seminormed space E. Then Φ is uniformly bounded.

Proof: Let $B_n = \{a: p(a) \leq n$ for all $p \varepsilon \Phi\}$ which implies that $\bigcup_{n=1}^{\infty} B_n = E$. By the Baire Category Theorem at least one $\bar{B}_k$

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Proof: Let $A_1, A_2, \ldots \varepsilon \Sigma$ a σ-field. Then $S-A_i \varepsilon \Sigma$ for $i=1,2,\ldots$

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Proof: Let $B_n = \{a: p(a) \leq n \text{ for all } p \varepsilon \Phi\}$ which implies that $\bigcup_{n=1}^{\infty} B_n = E$. By the Baire Category Theorem at least one $\bar{B}_k$

has non-empty interior. Now $B_k = \bigcap_{p\varepsilon\Phi} \{x: p(x)\leq k\}$ is closed and thus there is $r>0$ such that

$$N(b,r) \subsetneq B_k.$$

For $a\varepsilon E$ such that $||a|| \leq 1$, let $x = b + ra$ which means that $x\varepsilon N(b,r)$ and thus $p(x)\leq k$ for all $p\varepsilon\Phi$. Hence

$$p(a)=p([x-b]/r)\leq 1/r\ [p(x)+p(b)]\leq 2k/r \text{ for all } p\varepsilon\Phi.$$

Thus $||p|| \leq 2k/r$ for all $p\varepsilon\Phi$.

Q. E. D.

Theorem D (Baire Category Theorem) (Royden [8, p. 121]): A complete metric space is not the union of a countable collection of nowhere dense sets.

Theorem E (Hahn Decomposition Theorem) (Halmos [5, p. 121]): If μ is a measure, there is two disjoint sets A and B whose union is S, such that A is positive and B is negative with respect to μ.

Theorem F (Eberlein-Smulian Theorem) (Dunford and Schwartz [3, p. 430]): Let A be a subset of the Banach Space E. Then A is weakly sequentially compact if and only if A is weakly conditionally compact.

Theorem G (Hahn-Banach Theorem) (Wilansky [9, p. 65]): Let S be subspace of a linear space. Let p be a seminorm defined on the whole space and f a linear functional defined on S such that $|f(a)| \leq p(a)$ for all $a\varepsilon S$. Then there is an extension F of f which is a linear functional on the whole space and which satisfies $|F(a)| \leq p(a)$ for all a.

<u>Theorem</u> <u>H</u> (Radon-Nikodym Theorem) (Halmos [5, p. 128]): If (X,S,μ) is a totally σ-finite measure space and if a σ-finite measure ν on S is absolutely continuous with respect to μ, then there exists a finite measurable function f on X such that

$$\nu(E) = \int_E f\, d\mu$$

for every measurable set E.

<u>Theorem</u> <u>I</u> (Riesz Representation Theorem) (Royden[8, p. 103]): Let F be a bounded linear functional on L^p, $1 \leq p < \infty$. Then there is a function $g \in L^q$ such that $F(f) = \int f \cdot g$. Also $||F|| = ||g||_q$.

<u>Theorem</u> <u>J</u> (Egoroff's Theorem) (Royden[8, p. 60]): If $\{f_n\}$ is a sequence of measurable functions such that $f_n \to f$ μ-almost everywhere on a measurable set (of finite measure), then given $\eta > 0$, there is a subset $A \subseteq E$ with $\mu A < \eta$ such that

$$f_n \to f \text{ uniformly on } E-A.$$

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